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A NOTE ON THE POINCARÉ SERIES OF THE INVARIANTS OF TERNARY FORMS

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Abstract. Analogue of Springer's formula for the Poincaré series of the algebra invariants of ternary form is found.

1. Let $I_{n,d}$ be graded algebra of invariants of n-ary form of degree d:

$$I_{n,d} = (I_{n,d})_0 + (I_{n,d})_1 + \cdots + (I_{n,d})_k + \cdots,$$

and

$$P_{n,d}(T) = 1 + \dim((I_{n,d})_1)T + \cdots + \dim((I_{n,d})_k)T^k + \cdots$$

be its Poincaré series. By using Sylvester-Cayley formula, the series $P_{2,d}(T)$ for small d were calculated by Sylvester and Franclin in [1]. The explicit formula for calculation of the Poincare series was derived by Springer, see [3]. Namely,

$$P_{2,d}(z) = \sum_{0 \leq k < d/2} (-1)^k \psi_{d-2\,k} \left(\frac{(1-z^2) z^{k(k+1)}}{(1-z^2)(1-z^4)\dots(1-z^{2k})(1-z^2)(1-z^4)\dots(1-z^{2d-2j})} \right),$$

here

$$(\psi_n f)(z^n) = \frac{1}{n} \sum_{k=1}^{n} f(e^{\frac{2i\pi k}{n}} z),$$

for arbitrary rational function $f \in \mathbb{C}(t)$. By using the formula the Poicare series $P_{2,d}(T)$ for $d < 17$ was calculated in [4]. Also, by using a variation of the formula in [5] an algorithm for the computation of the Poincaré series is proposed and these series were calculated for even $d \leq 36$.

For the Poincare series of compact group G there exists the Molien–Weyl integral formula. In the case $G = SL(2, \mathbb{C})$ it can be written in the following form

$$P_{2,d}(T) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{1 - z^{-2}}{\prod_{k=1}^{n} (1 - T z^{d-2\,k})} \frac{dz}{z}, |z| < 1,$$

see [2], page 183. By calculation the integral, in [6] the series $P_{2,d}(T)$ was found for $d \leq 30$.

We know very little about the Poicare series $P_{3,d}(T)$ for the algebra of invariants of ternary form. In the paper [7] the series $P_{3,4}(T)$ was calculated. Also, in the paper [10] an analogue of Sylvester-Cayley formula was derived and listed several first terms of Poincare series $P_{3,d}(t)$ for small $d \leq 7$.

The aim of this paper is to derive an analogue of Springer's formula for the Poincare series of the algebra invariants of the ternary form.

2. We begin by short proving the Springer's formula for the Poincaré series of the algebra invariants of the binary form. Let us consider the $\mathbb{C}$-algebra $\mathbb{C}[[z]]$ of formal power series in z. For arbitrary $n \in \mathbb{N}$ define $\mathbb{C}$-linear function

$$\varphi_n : \mathbb{C}[[z]] \to \mathbb{C}[[z]]$$

in the following way

$$\varphi_n\big(z^m\big) := \begin{cases} z^{\frac{m}{n}}, \text{ if } m = 0 \mod n, \\ 0, \text{ if } m \neq 0 \mod n, \\ 1, \text{ for } m = 0. \end{cases}$$

Then for arbitrary series

$$A = a_0 + a_1 z + a_2 z^2 + \cdots,$$

we get

$$\varphi_n(A) = a_0 + a_n z + a_{2\,n} z^2 + \cdots + a_{s\,n} z^s + \cdots.$$

The lemma give us the explicit forms of the function φ_n, $n > 0$

Lemma 1. *For any $f \in \mathbb{C}[[z]]$ the following representations hold*

$$(i) \quad \varphi_n(f(z)) = \frac{1}{n}\sum_{k=1}^{n} f(z\, e^{\frac{2\pi i\, k}{n}})\Big|_{z^n=z};$$

$$(ii) \quad \varphi_n(f(z)) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(ze^{i\theta})}{1-e^{-i\,n\,\theta}} d\theta\Big|_{z^n=z}.$$

Proof. (i)

Put $\xi = e^{\frac{2\pi i}{n}}$. We have

$$\frac{1}{n}\sum_{k=1}^{n}\big(\xi^k z\big)^m = z^m \frac{1}{n}\sum_{k=1}^{n}\big(\xi^k\big)^m = \begin{cases} z^m, \text{ if } m = 0 \pmod n, \\ \\ 0, \text{ otherwise.} \end{cases}$$

It is follows that

$$\varphi_n(f(z))(z^m) = \begin{cases} z^{\frac{m}{n}} \text{ if } m = 0 \pmod n, \\ \\ 0, \text{ otherwise} \end{cases}$$

This construction is due to Simson, see [8] or [9], page 14.

(ii) Set $f(z) = \sum_{k=0}^{\infty} f_k z^k$. Then, taking into account that for integer n the integral $\displaystyle\int_0^{2\pi} e^{i\,n\theta} d\theta$ is equal to zero, we get

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{f(ze^{i\theta})}{1-e^{-i\,n\,\theta}} d\theta = \frac{1}{2\pi}\int_0^{2\pi} \sum_{k,s=0}^{\infty} f_k z^k e^{k\,i\theta} e^{-sn\theta} = \frac{1}{2\pi}\sum_{k=0}^{\infty} f_k z^k \sum_{s=0}^{\infty}\int_0^{2\pi} e^{(k-s\,n)i\theta} d\theta =$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\sum_{s=0}^{\infty} f_{n\,s} z^{n\,s} d\theta = \sum_{s=0}^{\infty} f_{n\,s} z^{n\,s}.$$

After replasing z^n by z we obtain the statement of the lemma.

□

As above, to work with formal power series in two letters, define $\mathbb{C}$-linear function $\Psi_{m,n} : \mathbb{C}[[t,z]] \to \mathbb{C}[[T]]$, $m, n \in \mathbb{N}$ by

$$\Psi_{n_1,n_2}\big(t^{m_1} z^{m_2}\big) = \begin{cases} T^s, \text{ if } \dfrac{m_1}{n_1} = \dfrac{m_2}{n_2} = s \in \mathbb{N}, \\ \\ 0, \text{ otherwise.} \end{cases}$$

If now

$$A = a_{0,0} + a_{1,0} t + a_{0,1} z + a_{2,0} t^2 + \cdots,$$

then, obviously,

$$\Psi_{n_1,n_2}(A) = a_{0,0} + a_{n_1,n_2}T + a_{2\,n_1,2\,n_2}T^2 + \cdots .$$

In some cases an calculation the functions $\Psi_{m,n}$ can be reduced to calculation of the functions such as φ. The following statement holds:

Lemma 2. *For any $R \in \mathbb{C}[[z]]$ and for $m, n, k \in \mathbb{N}$ we have*

$$\Psi_{m,n}\left(\frac{R}{1-t^m z^k}\right) = \begin{cases} \varphi_{n-k}(R) \text{ if } n \geq k, \\ \\ 0, \text{ if } n < k \end{cases}$$

Proof. Put $R = \sum_{j=0}^{\infty} f_j z^j$, $f_j \in \mathbb{C}$. Then for $k < n$ we get

$$\Psi_{m,n}\left(\frac{R}{1-t^m z^k}\right) = \Psi_{m,n}\Big(\sum_{j,s\geq 0} f_j z^j (t^m z^k)^s\Big) = \Psi_{m,n}\Big(\sum_{s\geq 0} f_{s(n-k)}(t^m z^n)^s\Big) = \sum_{s\geq 0} f_{s(n-k)}T^s.$$

On the other hand $\varphi_{n-k}(R) = \varphi_{n-k}\Big(\sum_{j=0}^{\infty} f_j z^j\Big) = \sum_{s\geq 0} f_{s(n-k)}T^s$. □

As in the proof of Lemma 1 we obtain that

$$\Psi_{1,1}\left(f(t,z)\right) = \frac{1}{2\pi}\int_0^{2\pi} f(t\,e^{-i\theta}, z\,e^{i\theta})\,d\theta.$$

The main idea of calculations of this work is that the Poincaré series $P_{2,d}(T)$ can be expressed in terms of functions Ψ. The following simple but important statement holds

Lemma 3.

$$P_{2,d}(T) = \Psi_{1,d}\big(f_d(t,z^2)\big).$$

where

$$f_d(t,z) = \frac{(1-z)}{(1-t)(1-tz)\dots(1-tz^d)} = \frac{1-z}{(t,z)_{d+1}}.$$

and $(a,q)_n = (1-a)(1-a\,q)\cdots(1-a\,q^{n-1})$ – q-shifted factorial.

Proof. For any $A \in \mathbb{C}[[t,z]]$ denote by $[t^n z^m]A$ the coefficients in $t^n z^m$. The Sylvester-Cayley formula implies that the dimension of the vector space $(I_{2,d})_n$ is equal to $[(tz^{d/2})^n]f_d(t,z)$. Then

$$P_{2,d}(T) = \sum_{n=0}^{\infty} \dim(I_{2,d})_n T^n = \sum_{n=0}^{\infty}\big([(tz^{d/2})^n]f_d(t,z)\big)T^n = \sum_{n=0}^{\infty}\big([(tz^d)^n]f_d(t,z^2)\big)T^n =$$

$$= \Psi_{1,d}(f_d(t,z^2)).$$

□

Now we can present simple proof of the Springer formula for the Poincaré series $P_{2,d}(T)$

Theorem 1 (Springer)**.**

$$P_{2,d}(T) = \sum_{0\leq k<d/2} \varphi_{d-2\,k}\left(\frac{(-1)^k z^{k(k+1)}(1-z^2)}{(z^2,z^2)_k\,(z^2,z^2)_{d-k}}\right),$$

Proof. Consider the partial fraction decomposition of the rational function $f_d(t,z)$:

$$f_d(t,z)=\sum_{k=0}^{d}\frac{R_k(z)}{1-tz^k}.$$

It is easy to see, that

$$\begin{aligned}R_k(z)&=\lim_{t\to z^{-k}}\big(f_d(t,z)(1-tz^k)\big)=\\&=\frac{1-z}{(1-z^{-k})\ldots(1-z^{-k}z^{k-1})\cdots(1-z^{-k}z^{k+1})\cdots(1-z^{-k}z^d)}=\\&=\frac{1-z}{z^{-(k+(k-1)+\ldots+1)}(z^k-1)(z^{k-1}-1)\ldots(z-1)\cdots(1-z)\cdots(1-z^{d-k})}=\\&=\frac{(-1)^k z^{\frac{k(k+1)}{2}}(1-z)}{(z,z)_k(z,z)_{d-k}}.\end{aligned}$$

Using the above lemmas we obtain

$$P_{2,d}=\Psi_{1,d}\big(f_d(t,z^2)\big)=\Psi_{1,d}\left(\sum_{k=0}^{n}\frac{R_k(z^2)}{1-tz^{2\,k}}\right)=\sum_{0\le k<d/2}\varphi_{d-2\,k}\left(\frac{(-1)^k z^{k(k+1)}(1-z^2)}{(z^2,z^2)_k(z^2,z^2)_{d-k}}\right).$$

□

3. Let us derive an formula for the Poicaré series of algebra of invariants of ternary form. In [10] was proved that the dimension of the vector space $(I_{3,d})_n$ is equal to $[\big(t(pq)^{\frac{d}{3}}\big)^n]f_d(t,p,q)$, where

$$f_d(t,p,q)=\frac{b_3(p,q)}{\prod\limits_{k+l\le d}(1-tp^kq^l)}=\frac{b_3(p,q)}{\prod\limits_{s=0}^{d}(tq^s,p)_{d+1-s}},$$

and $b_3(p,q)=1+p\,q+\dfrac{q^2}{p}-2\,q-q^2.$

On the ring $\mathbb{C}[[t,p,q,p^{-1},q^{-1}]]$ let us define the $\mathbb{C}$-linear functions Φ_{n_1,n_2,n_3} and $\hat{\Phi}_{n_1,n_2}$, $n_i\ge 0$ in the following way

$$\Phi_{n_1,n_2,n_3}\big(t^{m_1}p^{m_2}q^{m_3}\big)=\begin{cases}T^s,\ \text{if } \dfrac{m_1}{n_1}=\dfrac{m_2}{n_2}=\dfrac{m_3}{n_3}=s\in\mathbb{N}, m_i\ge 0,\\[2ex] 0,\ \text{otherwise.}\end{cases}$$

$$\hat{\Phi}_{n_1,n_2}\big(p^{m_1}q^{m_2}\big)=\begin{cases}T^s,\ \text{if } \dfrac{m_1}{n_1}=\dfrac{m_2}{n_2}=s\in\mathbb{N}, m_i\ge 0,\\[2ex] 0,\ \text{otherwise}\end{cases}$$

The following result may be proved in much the same way as Lemma 2 and Lemma 3 :

Lemma 4.

(*i*) *For arbitrary* $R \in \mathbb{C}[[p, q, p^{-1}, q^{-1}]]$ *and for any natural* m, n, k *we have:*

$$\Phi_{n_1,n_2,n_3}\left(\frac{R}{1 - t^{n_1}p^k q^l}\right) = \begin{cases} \hat{\Phi}_{n_2-k,n_3-l}(R) \text{ if } n_2 \geq k \text{ and } n_3 \geq l, \\ 0, \text{ if } n_2 < k \text{ or } n_3 < l. \end{cases};$$

(*ii*) $P_{3,d}(T) = \Psi_{1,d,d}\big(f_d(t, p^3, q^3)\big)$.

Now we are able to prove the analogue of Springer's formula for the Poincaré series of the algebra invariants of ternary form.

Theorem 2.

$$P_{3,d}(T) = \sum_{0 \leq k,j \leq [d/3]} \hat{\Phi}_{d-3\,k,d-3\,j}\left(\frac{(-1)^k p^{\frac{3k(k+1)}{2}} b_3(p^3, q^3)}{\left(\prod_{s=0, s\neq j}^{d} (q^{-3k}, p^{3(s-j)})_{d+1-s}\right)(p^3, p^3)_k\,(p^3, p^3)_{d-(k+j)}}\right).$$

Proof. Consider the partial fraction decomposition of the rational function $f_d(t, p, q)$:

$$f_d(t, p, q) = \sum_{k+j \leq d} \frac{R_{k,j}(p, q)}{1 - tp^k q^j}.$$

We have

$$R_{k,j}(p, q) = \lim_{t \to p^{-k}q^{-j}} \big(f_d(t, p, q)(1 - tp^k q^j)\big) = \lim_{t \to p^{-k}q^{-j}} \left(\frac{b_3(p, q)(1 - tp^k q^j)}{\left(\prod_{s\neq j}^{d} (tq^s, p)_{d+1-s}\right)(tq^j, p)_{d+1-j}}\right) =$$

$$= \frac{b_3(p, q)}{\left(\prod_{s\neq j}^{d} (p^{-k}q^{s-j}, p)_{d+1-s}\right)(1 - p^{-k}) \ldots (1 - p^{-1})(1 - p) \ldots (1 - p^{d-j-k})} =$$

$$= \frac{(-1)^k p^{\frac{k(k+1)}{2}} b_3(p, q)}{\left(\prod_{s\neq j}^{d} (p^{-k}q^{s-j}, p)_{d+1-s}\right)(p, p)_k (p, p)_{d-k-j}}.$$

The Lemma 4 now yields

$$P_{3,d}(T) = \Phi_{1,d,d}\left(\sum_{k+j \leq d} \frac{R_{k,j}(p^3, q^3)}{1 - tp^{3k}q^{3j}}\right) = \sum_{k+j \leq d} \hat{\Phi}_{d-3k,d-3j}\Big(R_{k,j}(p^3, q^3)\Big) =$$

$$= \sum_{0 \leq k,j \leq [d/3]} \hat{\Phi}_{d-3k,d-3j}\Big(R_{k,j}(p^3, q^3)\Big).$$

□

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